Variable Order Fractional Variational Calculus for Double Integrals*

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***Abstract*— We introduce three types of partial fractional operators of variable order. An integration by parts formula for partial fractional integrals of variable order and an extension of Green's theorem are proved. These results allow us to obtain a fractional Euler–Lagrange necessary optimality condition for variable order two-dimensional fractional variational problems.**

***Index Terms*— Variable order fractional calculus, fractional calculus of variations, Green's theorem, optimality conditions.**

I. INTRODUCTION

Fractional variational calculus is a mathematical discipline that consists in extremizing (minimizing or maximizing) functionals whose Lagrangians contain fractional integrals and derivatives. For the first link between calculus of variations and fractional calculus we should look back to the XIXth century. In 1823, Niels Heinrik Abel considered the problem (Abel's mechanical problem) of finding a curve, lying in a vertical plane, for which the time taken by a material point sliding without friction from the highest point to the lowest one, is destined function of height [1]. Abel's mechanical problem is a generalization of the tautochrone problem, which is part of the calculus of variations (and optimal control). Despite of this early example, fractional variational calculus became a research field only in the XXth century. The subject was initiated in 1996-1997 by Riewe, who derived Euler–Lagrange fractional differential equations and showed how non-conservative systems in mechanics can be described using fractional derivatives [32], [33]. Nowadays, the fractional calculus of variations and fractional optimal control are strongly developed (see, e.g., [2], [4], [5], [11], [14], [15], [16], [19], [22], [23], [24], [25]). For the state of the art, we refer the reader to the recent book [21].

In 1993, Samko and Ross investigated integrals and derivatives not of a constant but of variable order [34], [35], [37]. Afterwards, several pure mathematical and applicational papers contributed to the theory of variable order fractional calculus (see, e.g., [6], [10], [13], [20], [30], [31]). Here, our primary goal is to study problems of the calculus of variations with functionals given by two-dimensional definite integrals involving partial derivatives of variable fractional order. It should be mentioned that most results in fractional variational calculus are for single time, and that the literature regarding the multidimensional case is scarce: in [3] a fractional theory of the calculus of variations for multiple integrals is developed for Riemann–Liouville fractional derivatives and integrals in the sense of Jumarie; in [12] a Lagrangian structure for the Stokes equation, the fractional wave equation, the diffusion or fractional diffusion equations, are obtained using a fractional embedding theory; and in [28] fractional isoperimetric problems of calculus of variations with double integrals are considered. Here we develop a more general fractional theory of the calculus of variations for multiple integrals, where the fractional order is not a constant but a function.

The article is organized as follows. In Section II we give the definitions and basic properties of both ordinary and partial integrals and derivatives of variable fractional order. An extension of Green's theorem, to the variable fractional order, is then obtained in Section III. Section IV gives the proof of a necessary optimality condition for the two-dimensional fundamental problem of the calculus of variations. We finish with Section V of conclusions.

II. VARIABLE ORDER FRACTIONAL OPERATORS

In this section we introduce the notions of ordinary and partial fractional operators of variable order. Along the text L_1 denotes the class of Lebesgue integrable functions, AC the class of absolutely continuous functions, and by $\partial_i F$ we understand the partial derivative of a certain function F with respect to its ith argument.

Definition 1: Let $0<\alpha(t,\tau)<1$ for all $t,\tau\in[a,b]$, $f\in L_1[a,b]$, and Γ be the Gamma function, i.e.,

$$\Gamma(z)=\int_0^\infty \mathrm{e}^{-t}t^{z-1}dt.$$

Then,

$$ {}_aI_t^{\alpha(\cdot,\cdot)}f(t)=\int\limits_a^t \frac{1}{\Gamma(\alpha(t,\tau))}(t-\tau)^{\alpha(t,\tau)-1}f(\tau)d\tau \quad (t>a)$$

is called the left Riemann–Liouville integral of variable fractional order $\alpha(\cdot,\cdot)$, while

$$ {}_tI_b^{\alpha(\cdot,\cdot)}f(t)=\int\limits_t^b \frac{1}{\Gamma(\alpha(\tau,t))}(\tau-t)^{\alpha(\tau,t)-1}f(\tau)d\tau \quad (t<b)$$

denotes the right Riemann–Liouville integral of variable fractional order $\alpha(\cdot,\cdot)$.

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Example 2 ([37]): Let $\alpha(t,\tau)=\alpha(t)$ be a function depending only on variable t, $\frac{1}{n}<\alpha(t)<1$ for all $t\in[a,b]$ and a certain $n\in\mathbb{N}$ greater or equal than two, and $\gamma>-1$. Then,

$$ {}_aI_t^{\alpha(\cdot)}(t-a)^{\gamma}=\frac{\Gamma(\gamma+1)(t-a)^{\gamma+\alpha(t)}}{\Gamma(\gamma+\alpha(t)+1)}. $$

Definition 3: Let $0<\alpha(t,\tau)<1$ for all $t,\tau\in[a,b]$. If ${}_aI_t^{1-\alpha(\cdot,\cdot)}f\in AC[a,b]$, then the left Riemann–Liouville derivative of variable fractional order $\alpha(\cdot,\cdot)$ is defined by

$$ \begin{aligned} {}_aD_t^{\alpha(\cdot,\cdot)}f(t)&=\frac{d}{dt}{}_aI_t^{1-\alpha(\cdot,\cdot)}f(t)\\ &=\frac{d}{dt}\int_a^t\frac{1}{\Gamma(1-\alpha(t,\tau))}(t-\tau)^{-\alpha(t,\tau)}f(\tau)d\tau \quad (t>a), \end{aligned} $$

while the right Riemann–Liouville derivative of variable fractional order $\alpha(\cdot,\cdot)$ is defined for functions f such that ${}_tI_b^{1-\alpha(\cdot,\cdot)}f\in AC[a,b]$ by

$$ \begin{aligned} {}_tD_b^{\alpha(\cdot,\cdot)}f(t)&=-\frac{d}{dt}{}_tI_b^{1-\alpha(\cdot,\cdot)}f(t)\\ &=\frac{d}{dt}\int_t^b\frac{-1}{\Gamma(1-\alpha(\tau,t))}(\tau-t)^{-\alpha(\tau,t)}f(\tau)d\tau \quad (t<b). \end{aligned} $$

Definition 4: Let $0<\alpha(t,\tau)<1$ for all $t,\tau\in[a,b]$. If $f\in AC[a,b]$, then the left Caputo derivative of variable fractional order $\alpha(\cdot,\cdot)$ is defined by

$$ \begin{aligned} {}_a^CD_t^{\alpha(\cdot,\cdot)}f(t)&={}_aI_t^{1-\alpha(\cdot,\cdot)}\frac{d}{dt}f(t)\\ &=\int_a^t\frac{1}{\Gamma(1-\alpha(t,\tau))}(t-\tau)^{-\alpha(t,\tau)}\frac{d}{d\tau}f(\tau)d\tau \quad (t>a), \end{aligned} $$

while the right Caputo derivative of variable fractional order $\alpha(\cdot,\cdot)$ is given by

$$ \begin{aligned} {}_t^CD_b^{\alpha(\cdot,\cdot)}f(t)&=-{}_aI_t^{1-\alpha(\cdot,\cdot)}\frac{d}{dt}f(t)\\ &=\int_t^b\frac{-1}{\Gamma(1-\alpha(\tau,t))}(\tau-t)^{-\alpha(\tau,t)}\frac{d}{d\tau}f(\tau)d\tau \quad (t<b). \end{aligned} $$

Let $\Delta_n=[a_1,b_1]\times\cdots\times[a_n,b_n]$, $n\in\mathbb{N}$, be a subset of $\mathbb{R}^n$, $\mathbf{t}=(t_1,\ldots,t_n)\in\Delta_n$, and $\alpha_i(\cdot,\cdot):[a_i,b_i]\times[a_i,b_i]\to\mathbb{R}$ be such that $0<\alpha_i(t_i,\tau)<1$ for all $t_i,\tau\in[a_i,b_i]$, $i=1,\ldots,n$. Partial integrals and derivatives of variable fractional order are a natural generalization of the corresponding one-dimensional variable order fractional integrals and derivatives.

Definition 5: Let function $f=f(t_1,\ldots,t_n)$ be continuous on the set Δ_n. The left Riemann–Liouville partial integral of variable fractional order $\alpha_i(\cdot,\cdot)$, with respect to the ith variable t_i, is given by

$$ \begin{aligned} {}_{a_i}I_{t_i}^{\alpha_i(\cdot,\cdot)}f(\mathbf{t})&=\int_{a_i}^{t_i}\frac{1}{\Gamma(\alpha_i(t_i,\tau))}(t_i-\tau)^{\alpha_i(t_i,\tau)-1}\\ &\quad\times f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau \quad (t_i>a_i), \end{aligned} $$

while

$$ \begin{aligned} {}_{t_i}I_{b_i}^{\alpha_i(\cdot,\cdot)}f(\mathbf{t})&=\int_{t_i}^{b_i}\frac{1}{\Gamma(\alpha_i(\tau,t_i))}(\tau-t_i)^{\alpha_i(\tau,t_i)-1}\\ &\quad\times f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau \quad (t_i<b_i) \end{aligned} $$

denotes the right Riemann–Liouville partial integral of variable fractional order $\alpha_i(\cdot,\cdot)$ with respect to variable t_i.

Definition 6: If ${}_{a_i}I_{t_i}^{1-\alpha_i(\cdot,\cdot)}f\in C^1(\Delta_n)$, then the left Riemann–Liouville partial derivative of variable fractional order $\alpha_i(\cdot,\cdot)$, with respect to the ith variable t_i, is given by

$$ \begin{aligned} {}_{a_i}D_{t_i}^{\alpha_i(\cdot,\cdot)}f(\mathbf{t})&=\frac{\partial}{\partial t_i}{}_{a_i}I_{t_i}^{1-\alpha_i(\cdot,\cdot)}f(\mathbf{t})\\ &=\frac{\partial}{\partial t_i}\int_{a_i}^{t_i}\frac{1}{\Gamma(1-\alpha_i(t_i,\tau))}(t_i-\tau)^{-\alpha_i(t_i,\tau)}\\ &\quad\times f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau \quad (t_i>a_i), \end{aligned} $$

while the right Riemann–Liouville partial derivative of variable fractional order $\alpha_i(\cdot,\cdot)$, with respect to the ith variable t_i, is defined for functions f such that ${}_{t_i}I_{b_i}^{1-\alpha_i(\cdot,\cdot)}f\in C^1(\Delta_n)$ by

$$ \begin{aligned} {}_{t_i}D_{b_i}^{\alpha_i(\cdot,\cdot)}f(\mathbf{t})&=-\frac{\partial}{\partial t_i}{}_{t_i}I_{b_i}^{1-\alpha_i(\cdot,\cdot)}f(\mathbf{t})\\ &=\frac{\partial}{\partial t_i}\int_{t_i}^{b_i}\frac{-1}{\Gamma(1-\alpha_i(\tau,t_i))}(\tau-t_i)^{-\alpha_i(\tau,t_i)}\\ &\quad\times f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau \quad (t_i<b_i). \end{aligned} $$

Definition 7: Let $f\in C^1(\Delta_n)$. The left Caputo partial derivative of variable fractional order $\alpha_i(\cdot,\cdot)$, with respect to the ith variable t_i, is defined by

$$ \begin{aligned} {}_{a_i}^CD_{t_i}^{\alpha_i(\cdot,\cdot)}f(\mathbf{t})&={}_{a_i}I_{t_i}^{1-\alpha_i(\cdot,\cdot)}\frac{\partial}{\partial t_i}f(\mathbf{t})\\ &=\int_{a_i}^{t_i}\frac{1}{\Gamma(1-\alpha_i(t_i,\tau))}(t_i-\tau)^{-\alpha_i(t_i,\tau)}\\ &\quad\times\frac{\partial}{\partial\tau}f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau \quad (t_i>a_i), \end{aligned} $$

while the right Caputo partial derivative of variable fractional order $\alpha_i(\cdot,\cdot)$, with respect to the ith variable t_i, is given by

$$ \begin{aligned} {}_{t_i}^CD_{b_i}^{\alpha_i(\cdot,\cdot)}f(\mathbf{t})&=-{}_{t_i}I_{b_i}^{1-\alpha_i(\cdot,\cdot)}\frac{\partial}{\partial t_i}f(\mathbf{t})\\ &=\int_{t_i}^{b_i}\frac{-1}{\Gamma(1-\alpha_i(\tau,t_i))}(\tau-t_i)^{-\alpha_i(\tau,t_i)}\\ &\quad\times\frac{\partial}{\partial\tau}f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)d\tau \quad (t_i<b_i). \end{aligned} $$

Remark 8: In Definitions 5, 6 and 7, all the variables except t_i are kept fixed. That choice of fixed values determines a function $f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}:[a_i,b_i]\to\mathbb{R}$ of one variable t_i,

$$ f_{t_1,\ldots,t_{i-1},\ldots,t_{i+1},\ldots,t_n}(t_i)=f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n). $$

By Definitions 1, 3, 4, 5, 6, and 7, we have

$$
\begin{aligned}
&{}_{a_i}I_{t_i}^{\alpha_i(\cdot,\cdot)} f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}(t_i)\\
&\qquad = {}_{a_i}I_{t_i}^{\alpha_i(\cdot,\cdot)} f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n),
\end{aligned}
$$

$$
\begin{aligned}
&{}_{t_i}I_{b_i}^{\alpha_i(\cdot,\cdot)} f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}(t_i)\\
&\qquad = {}_{t_i}I_{b_i}^{\alpha_i(\cdot,\cdot)} f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n),
\end{aligned}
$$

$$
\begin{aligned}
&{}_{a_i}D_{t_i}^{\alpha_i(\cdot,\cdot)} f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}(t_i)\\
&\qquad = {}_{a_i}D_{t_i}^{\alpha_i(\cdot,\cdot)} f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n),
\end{aligned}
$$

$$
\begin{aligned}
&{}_{t_i}D_{b_i}^{\alpha_i(\cdot,\cdot)} f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}(t_i)\\
&\qquad = {}_{t_i}D_{b_i}^{\alpha_i(\cdot,\cdot)} f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n),
\end{aligned}
$$

$$
\begin{aligned}
&{}_{a_i}^{C}D_{t_i}^{\alpha_i(\cdot,\cdot)} f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}(t_i)\\
&\qquad = {}_{a_i}^{C}D_{t_i}^{\alpha_i(\cdot,\cdot)} f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n),
\end{aligned}
$$

$$
\begin{aligned}
&{}_{t_i}^{C}D_{b_i}^{\alpha_i(\cdot,\cdot)} f_{t_1,\ldots,t_{i-1},t_{i+1},\ldots,t_n}(t_i)\\
&\qquad = {}_{t_i}^{C}D_{b_i}^{\alpha_i(\cdot,\cdot)} f(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n).
\end{aligned}
$$

Thus, similarly to the integer order case, computation of partial derivatives of variable fractional order is reduced to the computation of one-variable derivatives of variable fractional order.

Remark 9: If $\alpha_i(\cdot,\cdot)$ is a constant function, then the partial operators of variable fractional order are reduced to corresponding partial integrals and derivatives of constant order. For more information on the classical fractional partial operators of constant order, we refer to [18], [29], [36].

III. Green's Theorem for Variable Order Fractional Operators

Green's theorem is useful in many fields of mathematics, physics, engineering, and fractional calculus [27]. We begin by proving a two-dimensional integration by parts formula for partial integrals of variable fractional order.

Theorem 10: Let $\frac{1}{l_i} < \alpha_i(t_i,\tau) < 1$ for all $t_i,\tau \in [a_i,b_i]$, where $l_i \in \mathbb{N}$, $i = 1,2$, are greater or equal than two. If $f,g,\eta_1,\eta_2 \in C(\Delta_2)$, then the partial integrals of variable fractional order satisfy the following identity:

$$
\begin{aligned}
&\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(\mathbf{t})\,{}_{a_1}I_{t_1}^{\alpha_1(\cdot,\cdot)}\eta_1(\mathbf{t}) + f(\mathbf{t})\,{}_{a_2}I_{t_2}^{\alpha_2(\cdot,\cdot)}\eta_2(\mathbf{t})\right]dt_2dt_1\\
&\quad = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[\eta_1(\mathbf{t})\,{}_{t_1}I_{b_1}^{\alpha_1(\cdot,\cdot)}g(\mathbf{t}) + \eta_2(\mathbf{t})\,{}_{t_2}I_{b_2}^{\alpha_2(\cdot,\cdot)}f(\mathbf{t})\right]dt_2dt_1.
\end{aligned}
$$

Proof: Define

$$
F_1(\mathbf{t},\tau) := \begin{cases} \left|\frac{(t_1-\tau)^{\alpha_1(t_1,\tau)-1}}{\Gamma(\alpha_1(t_1,\tau))}g(\mathbf{t})\eta_1(\tau,t_2)\right| & \text{if } \tau \le t_1\\ 0 & \text{if } \tau > t_1 \end{cases}
$$

for all $(\mathbf{t},\tau) \in [a_1,b_1]\times[a_2,b_2]\times[a_1,b_1]$, and

$$
F_2(\mathbf{t},\tau) := \begin{cases} \left|\frac{(t_2-\tau)^{\alpha_2(t_2,\tau)-1}}{\Gamma(\alpha_2(t_2,\tau))}f(\mathbf{t})\eta_2(t_2,\tau)\right| & \text{if } \tau \le t_2\\ 0 & \text{if } \tau > t_2 \end{cases}
$$

for all $(\mathbf{t},\tau) \in [a_1,b_1]\times[a_2,b_2]\times[a_2,b_2]$. Since f,g and η_i, $i=1,2$, are continuous functions on Δ_2, they are bounded on Δ_2, i.e., there exist positive real numbers C_1, C_2, C_3, $C_4 > 0$ such that

$$
|f(\mathbf{t})| \le C_1,\ \ |g(\mathbf{t})| \le C_2,\ \ |\eta_1(\mathbf{t})| \le C_3,\ \ |\eta_2(\mathbf{t})| \le C_4
$$

for all $\mathbf{t} \in \Delta_2$. Therefore,

$$
\begin{aligned}
&\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(\int_{a_1}^{b_1}F_1(\mathbf{t},\tau)d\tau + \int_{a_2}^{b_2}F_2(\mathbf{t},\tau)d\tau\right)dt_2\right)dt_1\\
&= \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(\int_{a_1}^{t_1}\left|\frac{(t_1-\tau)^{\alpha_1(t_1,\tau)-1}}{\Gamma(\alpha_1(t_1,\tau))}g(\mathbf{t})\eta_1(\tau,t_2)\right|d\tau\right.\right.\\
&\quad\left.\left. + \int_{a_2}^{t_2}\left|\frac{(t_2-\tau)^{\alpha_2(t_2,\tau)-1}}{\Gamma(\alpha_2(t_2,\tau))}f(\mathbf{t})\eta_2(t_1,\tau)\right|d\tau\right)dt_2\right)dt_1\\
&\le \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(C_2C_3\int_{a_1}^{t_1}\left|\frac{1}{\Gamma(\alpha_1(t_1,\tau))}(t_1-\tau)^{\alpha_1(t_1,\tau)-1}\right|d\tau\right.\right.\\
&\quad\left.\left. + C_1C_4\int_{a_2}^{t_2}\left|\frac{(t_2-\tau)^{\alpha_2(t_2,\tau)-1}}{\Gamma(\alpha_2(t_2,\tau))}\right|d\tau\right)dt_2\right)dt_1.
\end{aligned}
$$

Because $\frac{1}{l_i} < \alpha_i(t_i,\tau) < 1$, $i=1,2$,

1) $\ln(t_i-\tau) \ge 0$ and $(t_i-\tau)^{\alpha_i(t_i,\tau)-1} < 1$ for $t_i-\tau \ge 1$;
2) while $\ln(t_i-\tau) < 0$ and $(t_i-\tau)^{\alpha_i(t_i,\tau)-1} < (t_i-\tau)^{\frac{1}{l_i}-1}$ for $t_i-\tau < 1$.

Therefore,

$$
\begin{aligned}
&\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(C_2C_3\int_{a_1}^{t_1}\left|\frac{1}{\Gamma(\alpha_1(t_1,\tau))}(t_1-\tau)^{\alpha_1(t_1,\tau)-1}\right|d\tau\right.\right.\\
&\left.\left. + C_1C_4\int_{a_2}^{t_2}\left|\frac{1}{\Gamma(\alpha_2(t_2,\tau))}(t_2-\tau)^{\alpha_2(t_2,\tau)-1}\right|d\tau\right)dt_2\right)dt_1\\
&< \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(C_2C_3\left(\int_{a_1}^{t_1-1}\frac{1}{\Gamma(\alpha_1(t_1,\tau))}d\tau\right.\right.\right.\\
&\quad\left. + \int_{t_1-1}^{t_1}\frac{1}{\Gamma(\alpha_1(t_1,\tau))}(t_1-\tau)^{\frac{1}{l_1}-1}d\tau\right)\\
&\quad + C_1C_4\left(\int_{a_2}^{t_2-1}\frac{1}{\Gamma(\alpha_2(t_2,\tau))}d\tau\right.\\
&\quad\left.\left.\left. + \int_{t_2-1}^{t_2}\frac{1}{\Gamma(\alpha_2(t_2,\tau))}(t_2-\tau)^{\frac{1}{l_2}-1}d\tau\right)\right)dt_2\right)dt_1.
\end{aligned}
$$

Moreover, by inequality

$$
\Gamma(x+1) \ge \frac{x^2+1}{x+1},
$$

valid for $x \in [0,1]$ (see [17]), and the property

$$
\Gamma(x+1) = x\Gamma(x)
$$

of the Gamma function, one has

$$\int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(C_2C_3\left(\int_{a_1}^{t_1-1}\frac{1}{\Gamma(\alpha_1(t_1,\tau))}d\tau + \int_{t_1-1}^{t_1}\frac{1}{\Gamma(\alpha_1(t_1,\tau))}(t_1-\tau)^{\frac{1}{l_1}-1}d\tau\right) + C_1C_4\left(\int_{a_2}^{t_2-1}\frac{1}{\Gamma(\alpha_2(t_2,\tau))}d\tau + \int_{t_2-1}^{t_2}\frac{1}{\Gamma(\alpha_2(t_2,\tau))}(t_2-\tau)^{\frac{1}{l_2}-1}d\tau\right)\right)dt_2\right)dt_1$$
$$\leq \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(C_2C_3\left(\int_{a_1}^{t_1-1}\frac{\alpha_1^2(t_1,\tau)+\alpha_1(t_1,\tau)}{\alpha_1^2(t_1,\tau)+1}d\tau + \int_{t_1-1}^{t_1}\frac{\alpha_1^2(t_1,\tau)+\alpha_1(t_1,\tau)}{\alpha_1^2(t_1,\tau)+1}(t_1-\tau)^{\frac{1}{l_1}-1}d\tau\right) + C_1C_4\left(\int_{a_2}^{t_2-1}\frac{\alpha_2^2(t_2,\tau)+\alpha_2(t_2,\tau)}{\alpha_2^2(t_2,\tau)+1}d\tau + \int_{t_2-1}^{t_2}\frac{\alpha_2^2(t_2,\tau)+\alpha_2(t_2,\tau)}{\alpha_2^2(t_2,\tau)+1}(t_2-\tau)^{\frac{1}{l_2}-1}d\tau\right)\right)dt_2\right)dt_1$$
$$< \int_{a_1}^{b_1}\left(\int_{a_2}^{b_2}\left(C_2C_3\left(\int_{a_1}^{t_1-1}d\tau + \int_{t_1-1}^{t_1}(t_1-\tau)^{\frac{1}{l_1}-1}d\tau\right) + C_1C_4\left(\int_{a_2}^{t_2-1}d\tau + \int_{t_2-1}^{t_2}(t_2-\tau)^{\frac{1}{l_2}-1}d\tau\right)\right)dt_2\right)dt_1$$
$$< (b_2-a_2)(b_1-a_1)\left[C_2C_3\left(\frac{b_1+a_1}{2}-1-a_1+l_1\right) + C_1C_4\left(\frac{b_2+a_2}{2}-1-a_2+l_2\right)\right] < \infty.$$

Hence, we can use Fubini's theorem to change the order of integration:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(t_1,t_2)\,{}_{a_1}I_{t_1}^{\alpha_1(\cdot,\cdot)}\eta_1(t_1,t_2) + f(t_1,t_2)\,{}_{a_2}I_{t_2}^{\alpha_2(\cdot,\cdot)}\eta_2(t_1,t_2)\right]dt_2dt_1$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(t_1,t_2)\int_{a_1}^{t_1}\frac{1}{\Gamma(\alpha_1(t_1,\tau))}(t_1-\tau)^{\alpha_1(t_1,\tau)-1}\eta_1(\tau,t_2)d\tau + f(t_1,t_2)\int_{a_2}^{t_2}\frac{1}{\Gamma(\alpha_2(t_2,\tau))}(t_2-\tau)^{\alpha_2(t_2,\tau)-1}\eta_2(t_1,\tau)d\tau\right]dt_2dt_1$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[\eta_1(\tau,t_2)\int_{\tau}^{b_1}\frac{1}{\Gamma(\alpha_1(t_1,\tau))}(t_1-\tau)^{\alpha_1(t_1,\tau)-1}g(t_1,t_2)dt_1\right]dt_2d\tau + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[\eta_2(t_1,\tau)\int_{\tau}^{b_2}\frac{1}{\Gamma(\alpha_2(t_2,\tau))}(t_2-\tau)^{\alpha_2(t_2,\tau)-1}f(t_1,t_2)dt_2\right]d\tau dt_1$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta_1(\tau,t_2)\,{}_{\tau}I_{b_1}^{\alpha_1(\cdot,\cdot)}g(\tau,t_2)dt_2d\tau + \int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta_2(t_1,\tau)\,{}_{\tau}I_{b_2}^{\alpha_2(\cdot,\cdot)}f(t_1,\tau)d\tau dt_1.$$

■

We are now in conditions to state and prove the Green theorem for derivatives of variable fractional order.

Theorem 11: Let $0<\alpha_i(t_i,\tau)<1-\frac{1}{l_i}$ for all $t_i,\tau\in[a_i,b_i]$, where $l_i\in\mathbb{N}$, $i=1,2$, are greater or equal than two. If $f,g,\eta\in C^1(\Delta_2)$ and ${}_{t_1}I_{b_1}^{1-\alpha_1(\cdot,\cdot)}g, {}_{t_2}I_{b_2}^{1-\alpha_2(\cdot,\cdot)}f\in C^1(\Delta_2)$, then the following formula holds:

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(\mathbf{t})\,{}_{a_1}^{C}D_{t_1}^{\alpha_1(\cdot,\cdot)}\eta(\mathbf{t}) + f(\mathbf{t})\,{}_{a_2}^{C}D_{t_2}^{\alpha_2(\cdot,\cdot)}\eta(\mathbf{t})\right]dt_2dt_1$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta(\mathbf{t})\left[{}_{t_1}D_{b_1}^{\alpha_1(\cdot,\cdot)}g(\mathbf{t}) + {}_{t_2}D_{b_2}^{\alpha_2(\cdot,\cdot)}f(\mathbf{t})\right]dt_2dt_1 + \oint_{\partial\Delta_2}\eta(\mathbf{t})\left[{}_{t_1}I_{b_1}^{1-\alpha_1(\cdot,\cdot)}g(\mathbf{t})dt_2 - {}_{t_2}I_{b_2}^{1-\alpha_2(\cdot,\cdot)}f(\mathbf{t})dt_1\right].$$

Proof: By definition of Caputo partial derivative of variable fractional order, Theorem 10, and the standard Green's theorem, one has

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(\mathbf{t})\,{}_{a_1}^{C}D_{t_1}^{\alpha_1(\cdot,\cdot)}\eta(\mathbf{t}) + f(\mathbf{t})\,{}_{a_2}^{C}D_{t_2}^{\alpha_2(\cdot,\cdot)}\eta(\mathbf{t})\right]dt_2dt_1$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[g(\mathbf{t})\,{}_{a_1}I_{t_1}^{1-\alpha_1(\cdot,\cdot)}\frac{\partial}{\partial t_1}\eta(\mathbf{t}) + f(\mathbf{t})\,{}_{a_2}I_{t_2}^{1-\alpha_2(\cdot,\cdot)}\frac{\partial}{\partial t_2}\eta(\mathbf{t})\right]dt_2dt_1$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\left[\frac{\partial}{\partial t_1}\eta(\mathbf{t})\,{}_{t_1}I_{b_1}^{1-\alpha_1(\cdot,\cdot)}g(\mathbf{t}) + \frac{\partial}{\partial t_2}\eta(\mathbf{t})\,{}_{t_2}I_{b_2}^{1-\alpha_2(\cdot,\cdot)}f(\mathbf{t})\right]dt_2dt_1$$
$$= -\int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta(\mathbf{t})\left[\frac{\partial}{\partial t_1}\,{}_{t_1}I_{b_1}^{1-\alpha_1(\cdot,\cdot)}g(\mathbf{t}) + \frac{\partial}{\partial t_2}\,{}_{t_2}I_{b_2}^{1-\alpha_2(\cdot,\cdot)}f(\mathbf{t})\right]dt_2dt_1 + \oint_{\partial\Delta_2}\eta(\mathbf{t})\left[{}_{t_1}I_{b_1}^{1-\alpha_1(\cdot,\cdot)}g(\mathbf{t})dt_2 - {}_{t_2}I_{b_2}^{1-\alpha_2(\cdot,\cdot)}f(\mathbf{t})dt_1\right]$$
$$= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta(\mathbf{t})\left[{}_{t_1}D_{b_1}^{\alpha_1(\cdot,\cdot)}g(\mathbf{t}) + {}_{t_2}D_{b_2}^{\alpha_2(\cdot,\cdot)}f(\mathbf{t})\right]dt_2dt_1 + \oint_{\partial\Delta_2}\eta(\mathbf{t})\left[{}_{t_1}I_{b_1}^{1-\alpha_1(\cdot,\cdot)}g(\mathbf{t})dt_2 - {}_{t_2}I_{b_2}^{1-\alpha_2(\cdot,\cdot)}f(\mathbf{t})dt_1\right].$$

■

IV. VARIABLE ORDER FRACTIONAL CALCULUS OF VARIATIONS FOR DOUBLE INTEGRALS

Let $\alpha_i(t_i,\tau)$, $i=1,2$, satisfy the assumptions of Theorem 11. We consider the following problem:

Problem 12: Find a function $u = u(\mathbf{t})$ for which the fractional variational functional

$$\mathscr{J}[u] = \int_{a_1}^{b_1}\int_{a_2}^{b_2} L\left(\mathbf{t}, u(\mathbf{t}), {}^{C}_{a_1}D^{\alpha_1(\cdot,\cdot)}_{t_1}u(\mathbf{t}), {}^{C}_{a_2}D^{\alpha_2(\cdot,\cdot)}_{t_2}u(\mathbf{t})\right) dt_2dt_1$$

subject to the boundary condition

$$u(\mathbf{t})|_{\partial\Delta_2} = \psi(\mathbf{t}), \tag{1}$$

where $\psi : \partial\Delta_2 \to \mathbb{R}$ is a given function, attains an extremum. We assume that $L \in C^1\left(\Delta_2 \times \mathbb{R}^3;\mathbb{R}\right)$; $t \mapsto \partial_{i+2}L$ is continuously differentiable, has continuously differentiable integral ${}_{t_i}I^{1-\alpha_i(\cdot,\cdot)}_{b_i}$, and continuous derivative ${}_{t_i}D^{\alpha_i(\cdot,\cdot)}_{b_i}$, $i = 1,2$. For simplicity of notation, we introduce the following operator:

$$\{u,\alpha_1,\alpha_2\}(\mathbf{t}) := \left(\mathbf{t}, u(\mathbf{t}), {}^{C}_{a_1}D^{\alpha_1(\cdot,\cdot)}_{t_1}u(\mathbf{t}), {}^{C}_{a_2}D^{\alpha_2(\cdot,\cdot)}_{t_2}u(\mathbf{t})\right).$$

A typical example for the cost functional $\mathscr{J}$ appears when one considers the shape of a string during the course of the vibration (cf. [3]):

$$\mathscr{J}[u] = \int_t^x\int_0^L \left[\sigma(x)\left({}^{C}_{a_2}D^{\alpha_2(\cdot,\cdot)}_{x}u(x,t)\right)^2 - \tau\left({}^{C}_{a_1}D^{\alpha_1(\cdot,\cdot)}_{t}u(x,t)\right)^2\right] dxdt,$$

where τ is the constant tension and $\sigma(x)$ is the string density.

Definition 13: A continuously differentiable function u is said to be admissible for Problem 12 if ${}^{C}_{a_i}D^{\alpha_i(\cdot,\cdot)}_{t_i}u$ exist and are continuous on the rectangle Δ_2, $i = 1,2$, and u satisfies the boundary condition (1).

Theorem 14: If u is a solution to Problem 12, then u satisfies the Euler–Lagrange equation

$$\partial_2 L\{u,\alpha_1,\alpha_2\}(\mathbf{t}) + {}_{t_1}D^{\alpha_1(\cdot,\cdot)}_{b_1}\partial_3 L\{u,\alpha_1,\alpha_2\}(\mathbf{t}) + {}_{t_2}D^{\alpha_2(\cdot,\cdot)}_{b_2}\partial_4 L\{u,\alpha_1,\alpha_2\}(\mathbf{t}) = 0, \quad \mathbf{t} \in \Delta_2. \tag{2}$$

Proof: Suppose that u is an extremizer for $\mathscr{J}$. Consider $\eta \in C^1(\Delta_2;\mathbb{R})$ such that ${}^{C}_{a_i}D^{\alpha_i(\cdot,\cdot)}_{t_i}\eta \in C(\Delta_2;\mathbb{R})$, $i = 1,2$, and $\eta(\mathbf{t})|_{\partial\Delta_2} \equiv 0$. We imbed u in the one-parameter family of functions $\{\hat{u} = u + \varepsilon\eta : |\varepsilon| < \varepsilon_0, \varepsilon_0 > 0\}$. Define

$$J(\varepsilon) = \mathscr{J}[\hat{u}] = \int_{a_1}^{b_1}\int_{a_2}^{b_2} L\left(\mathbf{t}, \hat{u}(\mathbf{t}), {}^{C}_{a_1}D^{\alpha_1(\cdot,\cdot)}_{t_1}\hat{u}(\mathbf{t}), {}^{C}_{a_2}D^{\alpha_2(\cdot,\cdot)}_{t_2}\hat{u}(\mathbf{t})\right) dt_2dt_1.$$

Then, a necessary condition for u to be an extremizer for $\mathscr{J}$ is given by

$$\left.\frac{dJ}{d\varepsilon}\right|_{\varepsilon=0} = 0 \Leftrightarrow \int_{a_1}^{b_1}\int_{a_2}^{b_2}\left(\partial_2 L\{u,\alpha_1,\alpha_2\}(\mathbf{t})\cdot\eta(\mathbf{t}) + \partial_3 L\{u,\alpha_1,\alpha_2\}(\mathbf{t})\cdot{}^{C}_{a_1}D^{\alpha_1(\cdot,\cdot)}_{t_1}\eta(\mathbf{t}) + \partial_4 L\{u,\alpha_1,\alpha_2\}(\mathbf{t})\cdot{}^{C}_{a_2}D^{\alpha_2(\cdot,\cdot)}_{t_2}\eta(\mathbf{t})\right)dt_2dt_1 = 0.$$

By Theorem 11, and since $\eta(\mathbf{t})|_{\partial\Delta_2} \equiv 0$, one has

$$\begin{aligned}\int_{a_1}^{b_1}\int_{a_2}^{b_2}&\left(\partial_3 L\{u,\alpha_1,\alpha_2\}(\mathbf{t})\cdot{}^{C}_{a_1}D^{\alpha_1(\cdot,\cdot)}_{t_1}\eta(\mathbf{t}) + \partial_4 L\{u,\alpha_1,\alpha_2\}(\mathbf{t})\cdot{}^{C}_{a_2}D^{\alpha_2(\cdot,\cdot)}_{t_2}\eta(\mathbf{t})\right)dt_2dt_1\\ &= \int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta(\mathbf{t})\left({}_{t_1}D^{\alpha_1(\cdot,\cdot)}_{b_1}\partial_3 L\{u,\alpha_1,\alpha_2\}(\mathbf{t}) + {}_{t_2}D^{\alpha_2(\cdot,\cdot)}_{b_2}\partial_4 L\{u,\alpha_1,\alpha_2\}(\mathbf{t})\right)dt_2dt_1.\end{aligned}$$

Therefore,

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\eta(\mathbf{t})\left(\partial_2 L\{u,\alpha_1,\alpha_2\}(\mathbf{t}) + {}_{t_1}D^{\alpha_1(\cdot,\cdot)}_{b_1}\partial_3 L\{u,\alpha_1,\alpha_2\}(\mathbf{t}) + {}_{t_2}D^{\alpha_2(\cdot,\cdot)}_{b_2}\partial_4 L\{u,\alpha_1,\alpha_2\}(\mathbf{t})\right)dt_2dt_1 = 0.$$

Condition (2) follows from the fundamental lemma of the calculus of variations. ■

V. CONCLUSIONS

Recently, the variable order fractional calculus has provided new insights into rich applications in diverse fields such as physics, cyber-physical systems, signal processing, and mean field games [7], [8], [9], [38]. In this article a multidimensional integration by parts formula for partial integrals of variable fractional order (Theorem 10) and a Green type theorem with derivatives and integrals of variable fractional order (Theorem 11) are proved. These theorems are then used to obtain Euler–Lagrange type equations for the minimization of a functional involving derivatives of variable fractional order (Theorem 14). Our results generalize the recent one-dimensional theory of the fractional calculus of variations of variable order [26] to the two-dimensional case, i.e., for fractional variational problems with double integrals.

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